CW-complex as an adjuction space

정리 1 *(1) Let X be a CW-complex of dimension p and $\{e_\alpha\}$ be p-cells in X with a characteristic function $\varphi_\alpha : D^p_\alpha \to X$.*
$\Rightarrow X \cong \coprod D_\alpha \cup_f X^{p-1}, \quad f = \coprod \varphi_\alpha|_{\partial D_\alpha}$
(2) Conversely, Y is a CW-complex of dim p-1 and $f : \coprod \partial D^p_\alpha \to Y$
$\Rightarrow X = \coprod D_\alpha \cup_f Y$ *is a CW-complex with* $X^{p-1} = Y$.

증명
(1) Let $E = \coprod_\alpha D_\alpha \coprod X^{p-1}$.
Define $h : E \to X$ by $h = (\coprod \varphi_\alpha) \coprod (incl.)$.
It suffices to show that h is a quotient map.

Let $C \subset X$ with $h^{-1}(C)$ closed in E. Then

$$\Rightarrow \begin{cases} C \cap X^{p-1} = h^{-1}(C) \cap X^{p-1} \text{is closed in } X^{p-1} \\ h^{-1}(C) \cap D_\alpha \text{ is closed in } D_\alpha, \forall\alpha \text{ and so compact.} \end{cases}$$

$$\Rightarrow \begin{cases} C \cap \bar{e}_\beta = (C \cap X^{p-1}) \cap \bar{e}_\beta \text{ is closed in } \bar{e}_\beta \text{whenever } dim e_\beta \leq p-1 \\ C \cap \bar{e}_\alpha = h(h^{-1}(C) \cap D_\alpha) : \text{compact and so closed in } \bar{e}_\alpha. \end{cases}$$

$\Rightarrow C$ is closed in X.

$$(2) \begin{cases} Y : \text{Hausdorff} \\ (\coprod D_\alpha, \coprod \partial D_\alpha) : \text{collared pair} \\ \coprod D_\alpha \coprod Y \xrightarrow{q} X \end{cases}$$

$\Rightarrow X$: Hausdorff.
이제 CW-complex의 성질을 만족하는지 check하자. 우선 (1)은 당연하고,

(2) $\varphi_\alpha = q|_{D_\alpha}$로 정의하면, characteristic function이 된다. 왜냐하면, 앞에서 $q|_{D_\alpha - \partial D_\alpha}$가 homeomorphism 임을 알기 때문이다.

(3) $\bar{e}_\alpha = \varphi_\alpha(D_\alpha) = q(D_\alpha)$ and note that $q(\partial D_\alpha)$ is compact and hence meets only finitely many cells.

(4) Let $X = \{e_\beta\}$ and $Y = \{e_\gamma\}$.
$A \subset X$ with $A \cap \bar{e}_\beta$ closed in $\bar{e}_\beta, \forall\beta$

$$\Rightarrow \begin{cases} q^{-1}(A) \cap Y = A \cap Y \text{ is closed in } Y, \\ \quad \text{since } (A \cap Y) \cap \bar{e}_\gamma = A \cap \bar{e}_\gamma \text{ is closed in } \bar{e}_\gamma. \\ q^{-1}(A) \cap D_\alpha = q^{-1}(A \cap \bar{e}_\alpha) \cap D_\alpha : \text{closed in } D_\alpha \end{cases}$$

$\Rightarrow q^{-1}(A)$ is closed in $\coprod D_\alpha \coprod Y$.

$\Rightarrow A$ is closed.

□

정리 2 *(1) Let X be a CW-complex.*
Then X is a coherent union of $X^0 \subset X^1 \subset X^2 \subset \cdots$.
(2) Conversely, if X is a coherent union of CW-complexes $X_0 \subset X_1 \subset X_2 \subset \cdots$ with X_p, the p-skeleton of X_{p+1}, then X is a CW-complex with $X^p = X_p$.

증명 Easy from the definitions.**(숙제 10)** □